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# Lower CS-Closed Sets and Functions

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In this work, we introduce a class of convex sets, $LCSC^{\mathscr{F}}(X)$, of a locally convex separated and *not necessarily separable* topological vector space $X$. They are called the lower CS-closed sets. This class contains the CS-closed sets, satisfies the property $\operatorname{core}(C) = \operatorname{int}(C)$, $\forall C \in LCSC^{\mathscr{F}}(X)$ when $X$ is metrizable barrelled, and is stable under many operations. Among them, the projection and the denumerable intersection. We characterize the lower CS-closed functions (i.e., the functions who have a lower CS-closed epigraph) as marginal functions of CS-closed ones and show that they are very stable too. We establish an open mapping and a closed graph theorem for the lower CS-closed relations. Finally, we show that every real extended valued lower CS-closed function defined on a metrizable barrelled space is continuous on the interior of its domain. This result allows us to extend classical theorems of convex duality by replacing lower semicontinuous functions by lower CS-closed ones. More than that, it systematizes and extends some methods of convex analysis. © 1999 Academic Press

*Key Words:* convex analysis; CS-closed; duality; openness.

## 1. INTRODUCTION

Consider a convex set $C$ of a locally convex Hausdorff (separated) topological vector space $X$ (a l.c.s. space in the rest of the paper). We are interested in the property

$$\operatorname{core}(C) = \operatorname{int}(C), \tag{1}$$

where $\operatorname{core}(C)$ is the algebraic interior of $C$, defined as

$$\operatorname{core}(C) = \{x \in C \mid \forall u \in X,\ \exists \varepsilon > 0,\ x + \varepsilon[-u, u] \subset C\},$$

and $\operatorname{int}(C)$ denotes the interior of $C$.



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The reason why we are interested in the equality (1) is that this property (possibly used with the Hahn–Banach theorem) has many applications in convex analysis: continuity of convex functions, open mapping theorem, closed graph theorem, calculus of conjugates, subdifferential calculus, etc.

One always has $\mathrm{int}(C) \subset \mathrm{core}(C)$. If $\mathrm{int}(C) \neq \emptyset$ or $\mathrm{core}(C) = \emptyset$ then (1) holds; but it is well known that (1) fails in general. For example, take for $X$ the space of all continuous real functions defined on a compact topological space $K$ (with $K$ infinite) endowed with the pointwise convergence topology, and consider the set $C = \{f \in X \mid \forall t \in K,\ f(t) > 0\}$; thus $\mathrm{core}(C) = C$ but $\mathrm{int}(C) = \emptyset$.

In fact to ensure that the relation (1) holds, one needs some conditions on $C$ and/or on the whole space $X$.

If $X$ is a finite dimensional, any convex set of $X$ satisfies (1). Any closed convex set in a barrelled space satisfies (1). A larger class of convex subsets satisfying (1) is given by the CS-closed sets (whose definition is recalled below) in a metrizable barrelled space.

All the classes of sets considered above have one same default; they are not sufficiently stable: essentially, the projection of a closed convex (resp. a CS-closed set) is not necessarily closed (resp. CS-closed).

One approach to define a class of convex sets sufficiently stable and containing the closed convex sets is to add measurability assumptions. One can consult Kusraev [12] and references therein. But it seems that it needs the space $X$ to be *separable* (or to introduce a similar hypothesis).

The purpose of this paper is to introduce a class of convex sets, containing the CS-closed sets, namely the lower CS-closed sets, which satisfies (1), enjoys remarkable stability properties, and does not need the space $X$ to be separable.

In fact a lower CS-closed set of a l.c.s. space $X$ is just the projection on $X$ of a CS-closed set of a product space $X \times Y$ with $Y$ Fréchet.

This class of sets appears implicitly in former works. However, it seems that it has never been studied (neither defined!). One can see the class of lower CS-closed sets as the result of the systematization (and extension) of techniques used before in convex analysis concerning the problems evoked above.

We will see that any lower CS-closed set of a metrizable barrelled space satisfies (1).

As a direct consequence of the definition, the projection of a lower CS-closed set of a Fréchet space is lower CS-closed too. Moreover the sum of two lower CS-closed sets is always lower CS-closed. In particular the sum of two closed linear spaces is always lower CS-closed but may fail to be CS-closed.

Here we develop some applications concerning the continuity and the subdifferentiability of a lower CS-closed function (a function whose epigraph is lower CS-closed).

We observe that in practice many convex functions are lower CS-closed but not lower semicontinuous or CS-closed. Below, we prove that the sum, the max, the infimal convolution, the level sum, and the composite of two lower CS-closed functions are lower CS-closed. We characterize a lower CS-closed function as a marginal function of a CS-closed one.

As an application, we give a general duality result involving lower CS-closed perturbation functional. In this way, we obtain also some calculus rules about asymptotic functionals under mild assumptions.

## 2. LOWER CS-CLOSED SETS AND FUNCTIONS

### 2.1. *Preliminaries*: *Some Properties of CS-Closed Sets*

Let us first recall the definition of the CS-closed sets. Let $C$ be a subset of a l.c.s. space. By a convex series of elements of $C$, we mean a series of the form $\sum_{n \in \mathbb{N}} \lambda_n x_n$, where $(\forall n \in \mathbb{N},\ x_n \in C)$, $(\forall n \in \mathbb{N},\ \lambda_n \geq 0)$, and $\sum_{n=0}^{+\infty} \lambda_n = 1$. We say that $C$ is CS-closed (some authors say ideally convex) if it contains the sum of every convergent convex series of its elements. Note that any closed convex set is CS-closed. The CS-closed sets are always convex but not necessarily closed: for example, every open convex set is CS-closed. In the sequel we shall use some properties of the CS-closed sets we recall below.

$(\mathscr{H}_1)$ If $\{X_i\}_{i \in I}$ is a family of l.c.s. spaces and for each $i \in I$, $A_i \subset X_i$ is a CS-closed set then $\prod_{i \in I} A_i$ is a CS-closed set of $\prod_{i \in I} X_i$ endowed with the product topology.

$(\mathscr{H}_2)$ Every intersection of CS-closed sets is CS-closed.

If $A$ is a CS-closed set of a l.c.s. space $X$ and $M$ is a subspace of $X$ then $A \cap M$ is a CS-closed set of $M$ endowed with the induced topology.

One says that a function $f\colon X \to \overline{\mathbb{R}}$ is CS-closed if its epigraph, $\operatorname{epi} f$, defined as $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R},\ f(x) \leq r\}$, is a CS-closed subset of $X \times \mathbb{R}$.

Due to $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$, the indicator function of a subset $A$ of $X$, that is, the function $\delta_A$ defined by $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ if $x \notin A$, is a CS-closed function if and only if $A$ is CS-closed.

Note that every convex lower semicontinuous function is CS-closed.

One can refer to Jameson [11], Lifshits [15], Holmes [10], or Kusraev and Kutateladze [13] for further details about the CS-closed sets.

### 2.2. *Lower CS-Closed Sets and Functions*

Let $A$ be a subset of a product space $X \times Y$; we denote by $A_X$ the projection of $A$ on the space $X$. Recall that a Fréchet space is a locally convex topological vector space which is metrizable and complete.

Let us introduce now the classes of sets and functions we are interested in.

DEFINITION 2.1. A subset $C$ of a l.c.s. space $X$ is said to be lower CS-closed if there exists a Fréchet space $Y$ and a CS-closed set $A$ of $X \times Y$ such that $C = A_X$.

We denote by $LCSC^{\mathcal{F}}(X)$ the set of all lower CS-closed subsets of $X$.

Every CS-closed set is a lower CS-closed set but the converse is false: for example, we will see that the sum of two lower CS-closed sets is always lower CS-closed but may fail to be CS-closed.

Let us give an example of lower CS-closed set: take a CS-closed function $f\colon X \to \overline{\mathbb{R}}$; the domain of $f$, $\operatorname{dom} f = \{x \in X,\ f(x) < +\infty\}$, is a lower CS-closed set because it is the projection of the epigraph of $f$ ($\operatorname{dom} f = [\operatorname{epi} f]_X$).

We extend Definition 2.1 to functions and give a link between lower CS-closed sets and lower CS-closed functions.

DEFINITION 2.2. A function $f\colon X \to \overline{\mathbb{R}}$ defined on a l.c.s. space $X$ is said to be lower CS-closed if its epigraph is a lower CS-closed subset of $X \times \mathbb{R}$.

LEMMA 2.1. *A subset $C$ of a l.c.s. space $X$ is lower CS-closed if and only if $\delta_C$ is a lower CS-closed function.*

### 2.3. *Stability of Lower CS-Closed Sets*

The main motivation for introducing this class of sets and functions lies in its simplicity and its strong stability regarding various operations.

THEOREM 2.1. *Let $C$ be a lower CS-closed set of a product $X \times Y$ of l.c.s. spaces where $Y$ is a Fréchet space. Then $C_X$ is a lower CS-closed set.*

Another interesting fact is:

THEOREM 2.2. *Lower CS-closed sets still satisfy $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$ for denumerable families.*

*Proof.* We prove only that the intersection of a denumerable family of lower CS-closed sets is lower CS-closed.

Let $\{C_n\}_{n \in \mathbb{N}}$ be a family of lower CS-closed sets of a l.c.s. space $X$. For each $n \in \mathbb{N}$, there exists a Fréchet space $Y_n$ and a CS-closed subset $A_n$ of $X \times Y_n$ such that $C_n = [A_n]_X$. Set $Y = \prod_{n \in \mathbb{N}} Y_n$ and $\hat{A}_n = \{(x, (y_k)_{k \in \mathbb{N}}) \in X \times Y \mid (x, y_n) \in A_n\}$. Then $Y$ is a Fréchet space and $\hat{A}_n$ is CS-closed. So $\bigcap_{n \in \mathbb{N}} \hat{A}_n$ is CS-closed too. Using the fact that

$$\bigcap_{n \in \mathbb{N}} C_n = \left[ \bigcap_{n \in \mathbb{N}} \hat{A}_n \right]_X,$$

we obtain the result. ∎

One can deduce from these properties various corollaries. We identify a set-valued mapping with its graph.

COROLLARY 2.1. *If $A$ is a lower CS-closed set of a Fréchet space $X$ and $T$ is a lower CS-closed set of $X \times Y$ where $Y$ is a l.c.s. space, then $T(A)$ is a lower CS-closed set of $Y$.*

*Proof.* By definition, we have $T(A) = [T \cap A \times Y]_Y$. So by Theorems 2.2 and 2.1, $T(A)$ is lower CS-closed. ∎

COROLLARY 2.2. *Let $X$ be a l.c.s. space. The level sets of a lower CS-closed function $f: X \to \overline{\mathbb{R}}$, that are the subsets of $X$*

$$[f \leq r] = \{x \in X \mid f(x) \leq r\}, \quad [f < r] = \{x \in X \mid f(x) < r\}, \quad r \in \mathbb{R},$$

*are lower CS-closed sets of $X$.*

*Proof.* For each $r \in \mathbb{R}$,

$$\begin{aligned} [f \leq r] &= [\operatorname{epi} f \cap (X \times ]-\infty, r])]_X, \\ [f < r] &= [\operatorname{epi} f \cap (X \times ]-\infty, r[)]_X. \end{aligned}$$

∎

It is known that a linear subspace of a metrizable space is CS-closed if and only if it is closed. Moreover, since the sum of two closed linear subspaces of a Banach space may fail to be closed (see [6]), the sum of two CS-closed sets does not need to be CS-closed.

COROLLARY 2.3. *The sum of two lower CS-closed subsets of a Fréchet space is still lower CS-closed.*

*Proof.* The set $A + B$ is the image of the lower CS-closed subset $A \times B$ by the addition whose graph is a closed linear space. ∎

COROLLARY 2.4. *Let $M: X \to Y$ and $N: Y \to Z$ be two set-valued maps where $X, Y, Z$ are l.c.s. spaces with $Y$ Fréchet. If $M$ and $N$ are lower CS-closed, then $N \circ M$ is lower CS-closed.*

*Proof.* By definition, one has $N \circ M = [M \times Z \cap X \times N]_{X \times Z}$. ∎

Let us consider a last case. Let $X$ and $Z$ be two l.c.s. spaces, $A \subset X \times Z$ and $B \subset X \times Z$. Recall that the right partial sum of $A$ and $B$ is the set

$$\begin{aligned} A \dot{+} B = \{(x, z) \in X \times Z \mid \exists z_1 \in Z, \exists z_2 \in Z, (x, z_1) \in A, \\ (x, z_2) \in B, z = z_1 + z_2\}. \end{aligned}$$

COROLLARY 2.5. *The right partial sum of two lower CS-closed sets $A$ and $B$ of a space $X \times Z$ where $X$ is a l.c.s. space and $Z$ is a Fréchet space is still lower CS-closed.*

*Proof.* We set

$$T = \{(u, x, y, z) \in X \times Z \times Z \times Z,\ z = x + y\}$$
$$C = \{(u, x, y) \in X \times Z \times Z \mid (u, x) \in A,\ (u, y) \in B\}.$$

Then $T$ is a closed linear subspace and $C$ is lower CS-closed. Thus $A \dot{+} B = [T \cap C \times Z]_{X \times Z}$ is lower CS-closed too. ∎

## 2.4. *Characterization of Lower CS-Closed Functions*

In this section, we prove that a lower CS-closed function can be described as marginal of a CS-closed function. We first observe that a function is lower CS-closed if and only if its strict epigraph is lower CS-closed. Recall that the strict epigraph of a function $f\colon X \to \overline{\mathbb{R}}$ is given by $\mathrm{epi}_s f = \{(x, r) \in X \times \mathbb{R} \mid f(x) < r\}$.

PROPOSITION 2.1. *Let $X$ be a l.c.s. space and $f\colon X \to \overline{\mathbb{R}}$. Then $f$ is lower CS-closed if and only if its strict epigraph is lower CS-closed.*

*Proof.* We first observe that, for all $\varepsilon > 0$,

$$\mathrm{epi}_s f = \mathrm{epi}\, f \dot{+} (X \times ]0, +\infty[), \tag{2}$$
$$\mathrm{epi}_s(f - \varepsilon) = \mathrm{epi}_s f \dot{+} (X \times [-\varepsilon, +\infty[). \tag{3}$$

Now if $f$ is lower CS-closed then (2) and Corollary 2.5 tell us that $\mathrm{epi}_s f$ is lower CS-closed.

Conversely, if $\mathrm{epi}_s f$ is lower CS-closed then, using Corollary 2.5 once more, (3) tells us that $\mathrm{epi}_s(f - \varepsilon)$ is lower CS-closed for all $\varepsilon > 0$. On the other hand one has

$$\mathrm{epi}\, f = \bigcap_{\varepsilon > 0,\ \varepsilon \in \mathbb{Q}} \mathrm{epi}_s(f - \varepsilon).$$

Therefore $\mathrm{epi}\, f$ is lower CS-closed, which completes the proof. ∎

We are now in position to characterize the class of lower CS-closed functions.

THEOREM 2.3. *Let $X$ be a l.c.s. space. A function $f\colon X \to \overline{\mathbb{R}}$ is lower CS-closed if and only if there exists a Fréchet space $Y$ and a CS-closed function $F\colon X \times Y \to \overline{\mathbb{R}}$ such that for all $x \in X$, $f(x) = \inf_{y \in Y} F(x, y)$.*

*Proof.* If $\operatorname{epi} f$ is lower CS-closed then there exists a Fréchet space $Y$ and a CS-closed set $A$ of $X \times \mathbb{R} \times Y$ such that $\operatorname{epi} f = A_{X \times \mathbb{R}}$. We then have

$$f(x) = \inf_{r \in \mathbb{R}} r + \delta_{\operatorname{epi} f}(x, r) = \inf_{(y, r) \in Y \times \mathbb{R}} r + \delta_A(x, r, y).$$

Setting $F(x, (y, r)) = r + \delta_A(x, r, y)$, we get a CS-closed function. Moreover $f(x) = \inf_{(y, r) \in Y \times \mathbb{R}} F(x, (y, r))$ with $Y \times \mathbb{R}$ Fréchet.

Assume that there exists a Fréchet space $Y$ and a CS-closed function $F$: $X \times Y \to \overline{\mathbb{R}}$ such that $f(x) = \inf_{y \in Y} F(x, y)$. Then

$$\operatorname{epi}_s f = [\operatorname{epi}_s F]_{X \times \mathbb{R}}. \tag{4}$$

Moreover $Y$ is a Fréchet space and by Proposition 2.1, $\operatorname{epi}_s F$ is lower CS-closed. Now, using Theorem 2.1, (4) tells us that $\operatorname{epi}_s f$ is lower CS-closed. Finally, applying Proposition 2.1 once more, we get the announced result. ∎

### 2.5. *Stability of Lower CS-Closed Functions*

Due to the preceding characterization, we have the following result:

COROLLARY 2.6. *Let $F$: $X \times Z \to \overline{\mathbb{R}}$ be a lower CS-closed function where $X$ is a l.c.s. space and $Z$ is a Fréchet space. Then the marginal function $f$: $X \to \overline{\mathbb{R}}$, defined by $f(x) = \inf_{z \in Z} F(x, z)$, is lower CS-closed.*

It is known that the sum of two CS-closed functions having affine continuous minorants is CS-closed. The next proposition says, in particular, that the sum of two CS-closed functions is always lower CS-closed.

COROLLARY 2.7. *Let $X$ be a l.c.s. space, $f$: $X \to \overline{\mathbb{R}}$ and $g$: $X \to \overline{\mathbb{R}}$ two lower CS-closed functions. Then $\alpha f + \beta g$ is a lower CS-closed function for all $\alpha, \beta > 0$.*

*Proof.* Essentially due to the fact that $\operatorname{epi}(f + g) = \operatorname{epi} f \dotplus \operatorname{epi} g$ (we make the convention $(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty$)). ∎

Concerning the supremum of a family of lower CS-closed functions, we have the following result:

COROLLARY 2.8. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of lower CS-closed functions defined on a l.c.s. space $X$. Then $\sup_{n \in \mathbb{N}} f_n$ is a lower CS-closed function.*

*Proof.* One has

$$\operatorname{epi}\Big(\sup_{n \in \mathbb{N}} f_n\Big) = \bigcap_{n \in \mathbb{N}} \operatorname{epi} f_n.$$

Then by Theorem 2.2, $\sup_{n\in\mathbb{N}} f_n$ is lower CS-closed. ∎

The next result concerns the case of a lower CS-closed composite function:

COROLLARY 2.9. *Let $X$ be a l.c.s. space, let $Z$ be a Fréchet space preordered by a convex cone $Z_+$, let $h$ be a mapping defined on a convex set* $\operatorname{dom} h$ *of $X$ with values in $Z$ which is convex with respect to $Z_+$, and let $g$: $Z \to \overline{\mathbb{R}}$ be a convex function nondecreasing on $h(\operatorname{dom} h) + Z_+$ with respect to the preorder induced by $Z_+$.*

*Assume that* $\operatorname{epi} h = \{(x, z) \in X \times \mathbb{Z} \mid x \in \operatorname{dom} h,\ z - h(x) \in \mathbb{Z}_+\}$ *is a lower CS-closed set of $X \times Z$ and that $g$ is lower CS-closed.*

*Then the composite function defined by*

$$g \circ h(x) = \begin{cases} g(h(x)) & \text{if } x \in \operatorname{dom} h \\ +\infty & \text{otherwise} \end{cases}$$

*is lower CS-closed.*

*Proof.* Since $g$ is nondecreasing on $h(\operatorname{dom} h) + Z_+$, one has

$$g \circ h(x) = \inf_{z\in Z} g(z) + \delta_{\operatorname{epi} h}(x, z)$$

for all $x \in X$. Then, by Lemma 2.1 and Corollary 2.7, the function $(x, z) \to g(z) + \delta_{\operatorname{epi} h}(x, z)$ is lower CS-closed. Thus $g \circ h$ is lower CS-closed by Corollary 2.6. ∎

The infimal convolution (also called epigraphical sum) and the level sum of two lower CS-closed functions are still lower CS-closed.

COROLLARY 2.10. *Let $f$: $X \to \overline{\mathbb{R}}$ and $g$: $X \to \overline{\mathbb{R}}$ be two lower CS-closed functions defined on a Fréchet space $X$. Then the infimal convolution of $f$ and $g$, $f \,\square\, g$, defined by*

$$f \,\square\, g(x) = \inf_{u\in X} f(x - u) + g(u),$$

*and the level sum of $f$ and $g$, $f \,\triangle\, g$, defined by*

$$f \,\triangle\, g(x) = \inf_{u\in X} f(x - u) \vee g(u),$$

*where $a \vee b$ denotes the maximum of any extended real numbers $a$ and $b$, are lower CS-closed.*

*Proof.* Set $f_1(x, u) = f(x - u)$, $g_1(x, u) = g(u)$ for all $(x, u) \in X \times X$. Then, by Corollary 2.9, these functions are lower CS-closed and so is the function $f_1 + g_1$. Now we can conclude by applying the Corollary 2.6 to the function

$$f \,\square\, g(x) = \inf_{u \in X} f_1(x,u) + g_1(x,u).$$

The proof is similar for the level sum of two functions. ∎

## 2.6. *Openness of the Lower CS-Closed Sets*

As said in the Introduction, we are interested in the property

$$\operatorname{core}(C) = \operatorname{int}(C) \tag{5}$$

where $C$ is a convex set of a l.c.s. space $X$.

Recall that a barrelled topological vector space is a locally convex topological vector space in which every barrel (an absorbing, convex, closed, and balanced subset) is a neighborhood of the origin.

It is known that (5) is satisfied for closed convex sets of barrelled spaces. It follows that convex semiclosed sets ($C$ is semiclosed if $\operatorname{int}(\operatorname{cl}(C)) = \operatorname{int}(C)$) of barrelled spaces still satisfy (5). Moreover, it is also known (see [10, 17.B; 11]) that the CS-closed sets of metrizable topological vector spaces are semiclosed; therefore any CS-closed set of a metrizable barrelled space satisfies (5).

The next theorem says that property (5) is true for lower CS-closed sets. This is a consequence of an open mapping theorem of Rodriguez and Simons [23].

THEOREM 2.4. *Let $X$ be a metrizable barrelled space. Let $C$ be a lower CS-closed set of $X$. Then* $\operatorname{core}(C) = \operatorname{int}(C)$.

## 2.7. *Open Mapping and Lower CS-Closed Graph Theorems*

Let us first recall some definitions:

DEFINITION 2.3. Let $X$ and $Y$ be two l.c.s. spaces, let $M$: $X \to Y$ be a set-valued mapping, and let $(x, y) \in M$.

$M$ is said to be open at $(x, y)$ if, given any neighborhood $U$ of $x$, $M(U)$ is a neighborhood of $y$.

$M$ is said to be lower semicontinuous at $(x, y)$ if, given any neighborhood $V$ of $y$, $M^{-1}(V)$ is a neighborhood of $x$.

From Theorem 2.4, one can state the following result:

THEOREM 2.5. *Let $X$ be a Fréchet space, let $Y$ be a metrizable barrelled space, let $M$: $X \to Y$ be a set-valued mapping with lower CS-closed graph, and let $(x, y) \in M$. Suppose that $y \in \operatorname{core}(M(X))$. Then $M$ is open at $(x, y)$.*

We will need the following lemma:

LEMMA 2.2 [13, 1.2.8]. *Let X be a convex subset of a product of vector spaces* $X \times Y$. *Suppose that* $(x, y) \in A$ *and* $B$ *is a subset of* $Y$ *such that* $y \in \operatorname{core}(B)$. *Then* $\operatorname{core}(A_X) \subset \operatorname{core}([An(X \times B)]_X)$.

*Proof of Theorem* 2.5. Let $V$ be a neighborhood of $x$ and let $U$ be a closed neighborhood of $x$ contained in $V$. Then $x \in \operatorname{core}(U)$. Using the fact that $y \in \operatorname{core}(M(X))$, one can derive from Lemma 2.2 that $y \in \operatorname{core}(M(U))$. On the other hand $M(U)$ is lower CS-closed so $y \in \operatorname{int}(M(U)) \subset \operatorname{int}(M(V))$, which completes the proof. ∎

*Remark* 2.1. 1. In the above theorem, if $X$ is metrizable barrelled, $Y$ is Fréchet, and $x \in \operatorname{core}(M^{-1}(Y))$, then $M$ is lower semicontinuous at $(x, y)$.

2. The preceding theorem generalizes Theorem 1 of [20]. It fits in the same line as the results of Borwein [5], Jameson [11], Kusraev and Kutateladze [13], Rodriguez and Simons [23], and Ursescu [26].

We can derive directly from the previous theorem an interesting result in the linear case:

COROLLARY 2.11. *Let X be a metrizable barrelled space, let Y be a Fréchet space and let* $A\colon X \to Y$ *be a linear mapping with lower CS-closed graph. Then A is continuous.*

*Remark* 2.2. This corollary fits in the line of the results of Martineau and Schwartz on the borelian graph theorem (see [16, 17, 24]).

### 2.8. *Continuity and Subdifferentiability of the Lower CS-Closed Functions*

We can establish a result about the continuity of a lower CS-closed function. The following result generalizes a result which says that any finite valued lower semicontinuous convex function on a barrelled space is continuous on this space. We will need the following well-known result (see [9] for example):

PROPOSITION 2.2. *Let X be a l.c.s. space, let* $f\colon X \to \overline{\mathbb{R}}$ *be a convex function, and let* $z \in \operatorname{dom} f$. *Then f is continuous at z if and only if f is bounded above in a neighborhood of z.*

THEOREM 2.6. *Let X be a metrizable barrelled space and let* $f\colon X \to \overline{\mathbb{R}}$. *Suppose that for all* $r \in \mathbb{R}$, $[f \le r] = \{x \in X \mid f(x) \le r\}$ *is a lower CS-closed set. Then f is continuous at each point of* $\operatorname{core}(\operatorname{dom} f)$ *which is equal to* $\operatorname{int}(\operatorname{dom} f)$.

*Proof.* If $z \in \operatorname{core}(\operatorname{dom} f)$ and $r > f(z)$ then $z \in \operatorname{core}([f \le r])$ but $[f \le r]$ is lower CS-closed so, using Theorem 2.4, it is a neighborhood of $z$ and, due to Proposition 2.2, $f$ is continuous at $z$. ∎

One of the applications of the continuity of a convex function at a point is its subdifferentiability at this point. We put this result in a more precise form. Recall that if $f\colon X \to \overline{\mathbb{R}}$ is a convex function over a topological vector space $X$ and $z$ is such that $f(z) \in \mathbb{R}$ then the subdifferential of $f$ at $z$, denoted by $\partial f(z)$, is the following subset of the topological dual of $X$

$$\partial f(z) = \{x^* \in X^* : \langle x^*, x - z\rangle + f(z) \le f(x),\ \forall x \in X\}.$$

We say that $f$ is subdifferentiable at $z$ if $\partial f(z) \neq \emptyset$.

COROLLARY 2.12. *Let $f\colon X \to \overline{\mathbb{R}}$ be a lower CS-closed function defined on a l.c.s. space $X$. Let $z \in X$ such that $f(z) \in \mathbb{R}$ and*

$$\mathbb{R}^+[\operatorname{dom} f - z] \text{ is a metrizable barrelled space.} \tag{6}$$

*Then $f$ is subdifferentiable at $z$.*

*Proof.* Set $M = \mathbb{R}^+[\operatorname{dom} f - z]$ and $\varphi(x) = f(x + z)$ so that $\operatorname{dom} \varphi = \operatorname{dom} f - z$, $\operatorname{epi} \varphi = \operatorname{epi} f - (z, 0)$ and $\partial f(z) = \partial\varphi(0)$.

$0 \in \operatorname{core}(\operatorname{dom} \varphi|_M)$ and $\operatorname{epi} \varphi|_M = \operatorname{epi} \varphi$ is a lower CS-closed set so $\varphi|_M$ is continuous at the origin.

There exists (due to the Hahn–Banach theorem) $x^* \in \partial\varphi|_M(0)$. Now another application of the Hahn–Banach theorem implies the existence of $y^* \in Y^*$ such that $y^*|_M = x^*$ ($X$ is locally convex). Using the fact that $\operatorname{dom} \varphi = \operatorname{dom} \varphi|_M$, we have $y^* \in \partial\varphi(0)$ so $y^* \in \partial f(z)$. ∎

*Remark* 2.3. The condition (6) of the preceding corollary is very close to the Attouch–Brezis hypothesis (see Attouch and Brezis [1]).

## 3. APPLICATIONS

### 3.1. *The Use of Lower CS-Closed Functions in Convex Duality Theory*

Let us recall in a few words the setting of perturbational convex duality theory in the line of Rockafellar [22]. Let $U$ and $X$ be two Fréchet spaces, let $U^*$ and $X^*$ respectively be their topological dual spaces, let $\Phi\colon U \times X \to \overline{\mathbb{R}}$ be a given proper convex and lower semicontinuous function, and let $\bar{x}^* \in X^*$.

Consider the optimization problem

$$\text{minimize} \quad \Phi(0, x) - \langle \bar{x}^*, x\rangle, \qquad x \in X. \tag{$\mathscr{P}$}$$

The dual problem ($\mathscr{Q}$) associated with the primal problem ($\mathscr{P}$) is traditionally defined as

$$\text{maximize} \quad -\Phi^*(u^*, \bar{x}^*), \qquad u^* \in U^*. \tag{$\mathscr{Q}$}$$

Here $\Phi^*$ is the Legendre–Fenchel conjugate function of $\Phi$ defined on the product space $U^* \times X^*$ by

$$\Phi^*(u^*, x^*) = \sup_{(u,x) \in U \times X} \langle u^*, u \rangle + \langle x^*, x \rangle - \Phi(u, x)$$

for each $(u^*, x^*) \in U^* \times X^*$.

One says that the problem $(\mathscr{P})$ is stable if both problems $(\mathscr{P})$ and $(\mathscr{Q})$ have the same value, possibly $-\infty$, and if $(\mathscr{Q})$ admits an optimal solution. Indeed this is the case when the following condition is fulfilled (see, for instance, [25, Theorem 2.1] for Banach spaces):

(Q.C.1) $\mathbb{R}_+[\operatorname{dom} \Phi]_U$ is a closed vector subspace of $U$.

It is important to point out that the condition (Q.C.1) does not depend on $\bar{x}^*$ (see [4]). This result, which derives systematically from a slight extension of the Attouch–Brezis theorem on the conjugate of the sum of two convex proper lower semicontinuous functions, supposes essentially that the perturbational function $\Phi$ is lower semicontinuous.

However, as we shall see later on, in many cases the function $\Phi$ is *not lower semicontinuous* but *lower CS-closed*!

This observation is at the origin of the following result:

THEOREM 3.1. *Let $\Phi: U \times X \to \overline{\mathbb{R}}$ be a lower CS-closed function. Suppose that* (Q.C.1) *holds. Then*

$$-\infty \leq \max_{u^* \in U^*} - \Phi^*(u^*, x^*) = \inf_{x \in X} \Phi(0, x) - \langle x^*, x \rangle < +\infty \tag{7}$$

*for all $x^* \in X^*$. Moreover,*

$$\inf_{x \in X} \Phi(0, x) - \langle x^*, x \rangle = \inf_{x \in X} \Phi^{**}(0, x) - \langle x^*, x \rangle. \tag{8}$$

*Proof.* Set

$$p(u) = \inf_{x \in X} \Phi(u, x) - \langle x^*, x \rangle$$

for $u \in U$. Thus $\operatorname{dom} p = [\operatorname{dom} \Phi]_U$ and (Q.C.1) entails

$$\inf(\mathscr{P}) = p(0) < +\infty.$$

Moreover, one always has

$$p(0) \geq \sup_{u^* \in U^*} - \Phi^*(u^*, x^*).$$

So if $p(0) = -\infty$, (7) is proved.

In the following we suppose that $p(0) \in \mathbb{R}$. From Corollary 2.6, $p$ is lower CS-closed. Using the condition (Q.C.1) and Corollary 2.12, we obtain that $p$ is subdifferentiable at the origin. Therefore, there exists $\bar{u}^* \in U^*$ such that for all $u \in U$,

$$p(0) = \inf_{x \in X} \Phi(0, x) - \langle x^*, x \rangle \leq \inf_{x \in X} (\Phi(u, x) - \langle x^*, x \rangle) - \langle \bar{u}^*, u \rangle$$

so

$$\inf_{(x, u) \in X \times U} \Phi(u, x) - \langle x^*, x \rangle - \langle \bar{u}^*, u \rangle \geq \inf_{x \in X} \Phi(0, x) - \langle x^*, x \rangle.$$

As the opposite inequality obviously holds we get

$$-\Phi^*(\bar{u}^*, x^*) = \inf_{x \in X} \Phi(0, x) - \langle x^*, x \rangle,$$

which completes the proof of (7).

To prove (8) it suffices to observe that

$$\sup_{u^* \in U^*} -\Phi^*(u^*, x^*) = \sup_{u^* \in U^*} -\Phi^{***}(u^*, x^*) \leq \inf_{x \in X} \Phi^{**}(0, x) - \langle x^*, x \rangle$$

and

$$\inf_{x \in X} \Phi^{**}(0, x) - \langle x^*, x \rangle \leq \inf_{x \in X} \Phi(0, x) - \langle x^*, x \rangle = \max_{u^* \in U^*} -\Phi^*(u^*, x^*).$$

▮

As a corollary, let us quote another general duality result involving marginal lower CS-closed functions:

COROLLARY 3.1. *Let* $\Phi\colon U \times X \to \overline{\mathbb{R}}$ *be a lower CS-closed function. Suppose that* (Q.C.1) *holds. Then the convex function*

$$n\colon x^* \in X^* \mapsto n(x^*) = \inf_{u^* \in U^*} \Phi^*(u^*, x^*)$$

*is weak* lower semicontinuous, exact* (*i.e., the infimum above is reached*) *and does not take the value* $-\infty$.

*Proof.* By Theorem 3.1, $n$ is exact and for all $x^* \in X^*$ one has

$$n(x^*) = -\inf_{x \in X} \Phi(0, x) - \langle x^*, x \rangle.$$

In other words

$$n(x^*) = (\Phi(0, \cdot))^*(x^*)$$

for all $x^* \in X^*$. Consequently $n$ is weak* lower semicontinuous.

Let us prove that $n$ does not take the value $-\infty$. Suppose the contrary. Then

$$(\Phi(0,\cdot))^* = n^*(\cdot) = +\infty,$$

and, a fortiori,

$$\Phi(0,x) = +\infty \qquad \forall x \in X.$$

Hence $0 \notin [\operatorname{dom}\Phi]_U$, a contradiction with (Q.C.1). ∎

The corollary below extends the Rockafellar and Attouch-Brezis theorems (see also [23]) to lower CS-closed functions:

COROLLARY 3.2. *Let $f, g\colon X \to \overline{\mathbb{R}}$ be two lower CS-closed functions on the Fréchet space $X$ such that*

$$\mathbb{R}^+[\operatorname{dom} f - \operatorname{dom} g] \text{ is a closed linear subspace of } X.$$

*Then we have $(f+g)^* = f^* \,\square\, g^*$. In particular, the inf-convolution $f^* \,\square\, g^*$ is convex and weak* lower semicontinuous. Moreover, $f^* \,\square\, g^*$ is exact (at each point of $X^*$) and does not take the value $-\infty$.*

*Proof.* For $(u,x) \in X \times X$ let

$$\Phi(u,x) = f(u-x) + g(x).$$

Then, by Corollaries 2.9 and 2.7, $\Phi$ is lower CS-closed and

$$[\operatorname{dom}\Phi]_X = \operatorname{dom} g - \operatorname{dom} f.$$

On the other hand the Legendre–Fenchel conjugate of $\Phi$ is

$$\Phi^*(u^*, x^*) = f^*(-u^*) + g^*(u^* + x^*)$$

for all $(u^*, x^*) \in X^* \times X^*$. By Theorem 3.1, one thus has

$$n(\cdot) = \inf_{u^* \in X^*} \Phi^*(u^*, \cdot) = (f^* \,\square\, g^*)(\cdot).$$

Moreover $n$ is exact and

$$n(x^*) = -\inf_{x \in X} \Phi(0,x) - \langle x^*, x\rangle = (f+g)^*(x^*)$$

for all $x^* \in X^*$, so that the proof is complete. ∎

### 3.2. *On the Conjugate of the Sum of a Lower CS-Closed Function and a Lower CS-Closed Composite Function*

Assume that the Fréchet space $Z$ is equipped with a preorder relation induced by a convex cone $Z_+$ of $Z$: for $z_1, z_2 \in Z$ we set

$$z_1 \le z_2 \qquad \text{if } z_2 - z_1 \in Z_+.$$

Let $h$ be a function defined on a nonempty convex subset $\operatorname{dom} h$ of another Fréchet space $X$ and taking its values in $Z$. We denote by

$$\operatorname{epi} h = \{(x, z) \in \operatorname{dom} h \times Z \mid h(x) \leq z\}$$

the epigraph of $h$ and assume that $\operatorname{epi} h$ is convex.

Let us consider two extended-real-valued convex functions, $f\colon X \to \overline{\mathbb{R}}$, $g\colon Z \to \overline{\mathbb{R}}$, with $g$ nondecreasing on the subset $h(\operatorname{dom} h) + Z_+$ of $Z$. Assuming that $\operatorname{epi} h$ is closed, $f \in \Gamma_0(X)$, and $g \in \Gamma_0(Z)$, the expression of the conjugate function of the convex function $f + g \circ h$ has been explicated in [7] under the following condition:

(Q.C.2)

$\mathbb{R}_+[\operatorname{dom} g - h(\operatorname{dom} f \cap \operatorname{dom} g) - Z_+]$ is a closed vector subspace of $Z$.

By using Theorem 3.1 we intend to recapture and generalize this result assuming only the set $\operatorname{epi} h$ and the functions $f, g$ are lower $CS$-closed (see [19] for another generalization).

Let us observe that the convex function $f + g \circ h$ defined on $X$ by

$$f + g \circ h(x) = \begin{cases} f(x) + g(h(x)) & \text{if } x \in \operatorname{dom} h, \\ +\infty & \text{otherwise} \end{cases}$$

may be written under the form

$$f + g \circ h(x) = \Phi(0, x)$$

where $\Phi$ is defined on $Z \times X$ by

$$\Phi(u, x) = \begin{cases} \inf\limits_{z \in Z} f(x) + g(z) + \delta_{\operatorname{epi} h}(x, z - u) & \text{if } x \in \operatorname{dom} h, \\ +\infty & \text{otherwise.} \end{cases}$$

Now observe that the conjugate function of $f + g \circ h$ is given by

$$(f + g \circ h)^*(x^*) = -\inf_{x \in X} \Phi(0, x) - \langle x^*, x \rangle$$

for each $x^* \in X^*$.

Here again the function $\Phi$ is *not necessarily lower semicontinuous* but it is not difficult to show that $\Phi$ is lower CS-closed. Now observing that

$$[\operatorname{dom} \Phi]_U = \operatorname{dom} g - h(\operatorname{dom} h \cap \operatorname{dom} f) - Z_+,$$

one sees that condition (Q.C.1) is satisfied.

To apply Theorem 3.1 we need the conjugate function of $\Phi$. To this end, let us introduce the positive cone $Z_+^*$ of $Z^*$ consisting of all the continuous positive linear forms $u^* \in Z_+^*$, that is,

$$\langle u^*, u\rangle \geq 0, \qquad \forall u \in Z_+.$$

In what follows, we extend the functions $u^* \circ h$ to the whole space $X$ by setting $(u^* \circ h)(x) = +\infty$ if $x \notin \operatorname{dom} h$. Hence we have the following result:

LEMMA 3.1. *For each $(u^*, x^*) \in Z^* \times X^*$, we have*

$$\Phi^*(u^*, x^*) = \begin{cases} (f + u^* \circ g)(x^*) + g^*(u^*) & \text{if } u^* \in Z_+^*, \\ +\infty & \text{otherwise}. \end{cases}$$

*Now, we derive the expression of the conjugate function of $f + g \circ h$, generalizing* [7, *Proposition* 4.11]:

COROLLARY 3.3. *Let $X, Z$ be two Fréchet spaces, $Z$ being equipped with a partial order induced by a convex cone $Z_+$, let $f$ and $g$ be two lower CS-closed functions on $X$ and $Z$ respectively, and let $h$ be a mapping defined on a convex subset* $\operatorname{dom} h$ *of $X$ with values in $Z$ and such that* $\operatorname{epi} h$ *is lower CS-closed. Assume that $g$ is nondecreasing on the subset $h(\operatorname{dom} h) + Z_+$ and that* (Q.C.2) *holds. Then we have*

$$(f + g \circ h)^*(x^*) = \min_{u^* \in Z_+^*} g^*(u^*) + (f + u^* \circ h)^*(x^*)$$

*for each $x^* \in X^*$.*

*Proof.* Applying Theorem 3.1 to the function $\Phi$, we can get the announced result. ∎

### 3.3. *Asymptotic calculus*

Let $U, X$ be two Fréchet spaces, $G$ a convex proper weak* lower semicontinuous function on $U^* \times X^*$, and $g(x^*) = \inf_{u^* \in U^*} G(u^*, x^*)$. Suppose that

(Q.C.3) $\qquad \mathbb{R}_+[\operatorname{dom} G^*]_U$ is a closed vector subspace of $U$,

where the conjugate function $G^*$ of $G$ is considered on the product space $U \times X$ and not $U^{**} \times X^{**}$.

By Corollary 3.1 apply to $\Phi = G^*$, $g$ is convex weak* lower semicontinuous, exact, and does not take the value $-\infty$. Furthermore $g^*(\cdot) = G^*(0, \cdot)$ so that, $G^*$ being proper, $g$ is not identically $+\infty$.

It follows that the asymptotic functional $g_\infty$ of $g$ coincides with the support function of the effective domain of $g^*$, i.e., for each $x^* \in X^*$, we have (see [14])

$$g_\infty(x^*) = \sup_{x \in \operatorname{dom} g^*} \langle x^*, x\rangle.$$

On the other hand

$$g^*(x) = G^*(0, x), \qquad \forall x \in X;$$

we then have

$$g_\infty(x) = \sup\langle x^*, x\rangle \mid G^*(0, x) < +\infty,$$

or, equivalently,

$$-g_\infty(x) = \inf_{x \in X} \delta_{\operatorname{dom} G^*}(0, x) - \langle x^*, x\rangle.$$

Of course $\delta_{\operatorname{dom} G^*}$ is *not necessarily lower semicontinuous nor CS-closed*. But $\delta_{\operatorname{dom} G^*}$ *is actually lower CS-closed*. Now we can state:

THEOREM 3.2. *Let $G \in \Gamma_0(U^* \times X^*)$ and let $g\colon X^* \to \overline{\mathbb{R}}$ be defined by*

$$g(x^*) = \inf_{u^* \in U^*} G(u^*, x^*).$$

*Suppose that* (Q.C.3) *holds. Then g is convex, proper, weak* l.s.c., exact and one has*

$$g_\infty(x^*) = \min_{u^* \in U^*} G_\infty(u^*, x^*) \tag{9}$$

*for all $x^* \in X^*$.*

*Proof.* We only need to prove (9).

Applying Theorem 3.1 to $\Phi = \delta_{\operatorname{dom} G^*}$, we then have

$$\max_{u^* \in U^*} -\delta^*_{\operatorname{dom} G^*}(u^*, x^*) = \inf_{x \in X} \delta_{\operatorname{dom} G^*}(0, x) - \langle x^*, x\rangle = -g_\infty(x) < +\infty$$

but $\delta^*_{\operatorname{dom} G^*} = G_\infty$ and the proof is complete. ∎

*Remark* 3.1. The formula (9) was established (see [29]) in the setting of locally convex space under some relative compactness assumptions which we do not need here (see also [3]).

As an example, let us consider the inf-convolution of two convex proper weak* lower semicontinuous functions $\varphi$ and $\psi$ defined on $X^*$. We know that if

(Q.C.4)

$$\mathbb{R}_+(\operatorname{dom} \varphi^* - \operatorname{dom} \psi^*) \text{ is a closed vector subspace of } X$$

then the inf-convolution $\varphi \,\square\, \psi$ defined by

$$x^* \in X^* \to (\varphi \,\square\, \psi)(x^*) = \inf_{u^* \in U^*} \varphi(x^* - u^*) + \psi(u^*)$$

is a convex proper weak* lower semicontinuous function (see [1]).

The formula

$$(\varphi \,\square\, \psi)_\infty = \varphi_\infty \,\square\, \psi_\infty$$

was established in the hilbertian setting in [2], by means of the asymptotic function associated with a monotone maximal operator, under the hypothesis

$$\operatorname{int}(\operatorname{dom} \partial\varphi^*) \cap \operatorname{dom} \partial\psi^* \neq \emptyset.$$

In fact we have the following result which allows us to weaken sensibly the above condition

COROLLARY 3.4. *Let* $\varphi, \psi \in \Gamma_0(X^*)$. *Suppose that* (Q.C.4) *holds*. *Then*

$$(\varphi \,\square\, \psi)_\infty = \varphi_\infty \,\square\, \psi_\infty.$$

*Proof.* Let

$$G(u^*, x^*) := \varphi(x^* - u^*) + \psi(u^*)$$

for all $u^*, x^*$ in $X^*$. Then we have

$$G^*(u, x) = \varphi^*(x) + \psi^*(u + x)$$

for all $u, x$ in $X$. It follows that

$$[\operatorname{dom} G^*]_U = \operatorname{dom} \psi^* - \operatorname{dom} \varphi^* \quad \text{and}$$

$$G_\infty(u^*, x^*) = \varphi_\infty(x^* - u^*) + \psi_\infty(u^*).$$

From Theorem 3.2 we obtain the announced result. ∎

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